Last time:

Prop: $(K, |-|)$ non-arch. valued, $\mathcal{O}_K \subseteq K$

$\mathcal{O}_K = \{x \in K \mid |x| \leq 1\}$

⚠ subring

1) $\mathcal{O}_K$ is integrally closed, local ring with maximal ideal $\mathfrak{m}_K = \{x \in K \mid |x| < 1\}$, & each fin. gen. ideal $I \subseteq \mathcal{O}_K$ is principal and of the form

$$\{x \in K \mid |x| \leq r\}, \quad r := \max(|y| \mid y \in I)$$

2) The subspace top. on $\mathcal{O}_K$ (w.r.t. the metric top. on K) is (x)-adic for each $x \in \mathfrak{m}_K \setminus \{0\}$, $\mathcal{O}_K \subseteq K$ is open & closed,

$K = \mathcal{O}_K[\frac{1}{x}] \quad \forall x \in \mathfrak{m}_K \setminus \{0\}$

i.e. $\underset{\neq 0}{x}$ top. nilpotent

& $\hat{K} = \hat{\mathcal{O}}_K[\frac{1}{x}]$, where

$\hat{\mathcal{O}}_K$ (x)-adic completion of $\mathcal{O}_K$, x top. nilp, $x \neq 0$

& $\mathcal{O}_{\hat{K}}$ is the closure of $\mathcal{O}_K$ in $\hat{K}$

For 2) we are left to prove

La: $(K, |-|)$ non-arch. valued,

$x \in \mathfrak{m}_K \setminus \{0\}$

$$\Rightarrow \quad \mathcal{O}_K / x \cdot \mathcal{O}_K \xrightarrow{\sim} \mathcal{O}_{\hat{K}} / x \cdot \mathcal{O}_{\hat{K}}$$

Prf of La: Let $\varphi : \mathcal{O}_K / x\mathcal{O}_K \to \mathcal{O}_{\hat{K}} / x\mathcal{O}_{\hat{K}}$ be the canonical morph.

1) φ surj.

Pick $\eta \in \mathcal{O}_{\hat{K}}$

Know: $K \subseteq \hat{K}$ is dense

$\Rightarrow \exists z \in K$, s.t.

$z - \eta \in x \cdot \mathcal{O}_{\hat{K}}$ (as $x \cdot \mathcal{O}_{\hat{K}}$ is open in $\hat{K}$)

$$\Rightarrow z \in y + x \cdot \mathcal{O}_{\hat K} \subseteq \mathcal{O}_{\hat K}$$

$$\Rightarrow z \in \underbrace{K \cap \mathcal{O}_{\hat K}}_{= \{t \in K \mid |t|_{\hat K} \le 1\}} = \mathcal{O}_K$$

$$|\cdot|_{\hat K} = |\cdot|_K \text{ on } K$$

$$\Rightarrow \varphi(z + x \cdot \mathcal{O}_K) = y + x \cdot \mathcal{O}_{\hat K}$$

$\Rightarrow \varphi$ surj.

2) Let $y \in \mathcal{O}_K$, s.t. $\varphi(y) = 0$

$$\Rightarrow \underbrace{|y|_{\hat K}}_{=|y|_K} \le \underbrace{|x|_{\hat K}}_{=|x|_K} \Rightarrow y \in \underbrace{x \cdot \mathcal{O}_K}_{=\{t \in K \mid |t| \le |x|\}}$$

□ of the la

Top. on K:

$V \subseteq K$ is open iff $\forall x \in V$ ex. some $r > 0$, s.t.

$$\{y \in K \mid |x - y| < r\} \subseteq V$$

$(\Leftarrow)$ $V \subseteq K$ is open iff $\forall\, x \in V$ ex. some $z \in \mathcal{O}_K \setminus \{0\}$, s.t.

$$\underbrace{x + z \cdot \mathcal{O}_K}_{\shortparallel} \subseteq V$$

$$\{ y \in K \mid |y - x| \leq \underset{\substack{\neq \\ 0}}{|z|} \}$$

Remains to prove:

3) If $|\cdot|$ discrete, then m_K principal

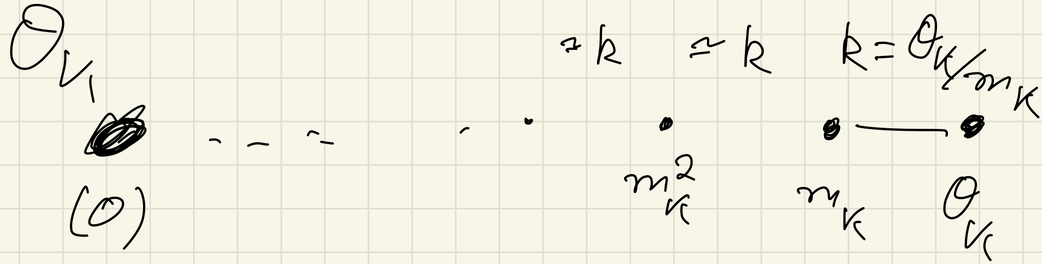
cf prop

$$\{ I \subseteq \mathcal{O}_K \text{ non-zero} \} \overset{1:1}{\longleftrightarrow} \mathbb{N}_{\geq 0}$$
$$m_K^n \longleftarrow n$$

and $\operatorname{Spec} \mathcal{O}_K = \{(0), m_K\}$

In particular, $\mathcal{O}_K$ is a local PID,

& $\{\text{non-zero fract. ideals}\} \overset{1:1}{\longleftrightarrow} \mathbb{Z}$
$$m_K^n \longleftarrow n$$



E.g.: $\mathcal{O}_K = k[[z]] \subseteq K = k((z))$

$$\begin{array}{cc} k[[z]] & \\ \cup\!\! \mid & k \\ (z) & \\ \cup\!\! \mid & k \\ (z^2) & \\ \vdots & \\ (0) & \end{array} \qquad\qquad \begin{array}{cc} \mathbb{Z}_p & \\ \cup\!\! \mid & \mathbb{F}_p \\ (p) & \\ \cup\!\! \mid & \mathbb{F}_p \\ (p^2) & \\ \cup\!\! \mid & \\ \vdots & \\ (0) & \end{array}$$

For 3):

Suff. to prove:

$|K^\times|$ discrete in $\mathbb{R}_{>0}$

implies that $\mathcal{O}_K$ noetherian

But from 1)

$$0 \neq I \subseteq \mathcal{O}_K$$

$$\Rightarrow \quad I = \bigcup_{y \in I} \{x \in \mathcal{O}_K \mid |x| \leq |y|\}$$

$$= \{x \in \mathcal{O}_K \mid |x| \leq r\} \text{ with}$$

$$r = \max\{|y| \mid y \in I\}$$

$|K^\times|$ discrete in $\mathbb{R}_{>0}$ (attained)

$\Rightarrow$ I principal $\square$

⚠ Discrete valuation rings are important. Let R be any ring.

TFAE:

1) $R = \mathcal{O}_K$ for $(K, |\cdot|)$ a discr. non-arch. valued field

2) R local Dedekind domain

3) R local, int. closed, noetherian
with max. ideal principal

Def: $(K, |-|)$ non-arch valued

1) $k = \mathcal{O}_K / m_K$ residue field of
$(K, |-|)$ (or $\mathcal{O}_K$, K)

2) any $x \in m_K \setminus \{0\}$ is called a
pseudo-uniformizer.

If $|-|$ discrete & $(x) = m_K$

$\Rightarrow$ x is called a uniformizer

Equiv (if $|-|$ discrete):
$v(x) = 1$, where $\quad v(x) = -\log_q |-|$

$v : K \to \mathbb{Z} \cup \{\infty\}$ is the
normalized add. valuation of K

Equiv: $|x| = \max \{ |y| \mid y \in m_K \}$

Prop: K field, $|-|_1, |-|_2$ two non-arch. norms

TFAE:

1) $|-|_1 \sim |-|_2$, i.e. $|-|_1 = |-|_2^c$, $c > 0$

2) The valuation rings for $|-|_1, |-|_2$ agree, i.e.
$\{x \in K \mid |x|_1 \leq 1\} = \{x \in K \mid |x|_2 \leq 1\}$

3) The metric topologies of $|-|_1$ & $|-|_2$ agree

Prf: 1) $\Rightarrow$ 2): ✓

2) $\Rightarrow$ 1): Set $\mathcal{O}_{v_1}$ as the valuation ring for $|-|_1$ (& $|-|_2$)

Pick $b \in K$, $|b|_1 > 1$

(if b doesn't exist $\Rightarrow \mathcal{O}_K = K$ & $|-|_1, |-|_2$ are trivial)

$\exists c > 0$, s.t. $|b|_2^c = |b|_1$

$(\text{as } |b|_2 > 1, \text{ bec. } b \notin \mathcal{O}_K)$

$\Rightarrow$ wlog $|b|_2 = |b|_1$

Have to see $|x|_2 = |x|_1 \ \forall x \in K$

Let $x \in K \setminus \{0\}$

$\Rightarrow \exists \varrho \in \mathbb{R}$, s.t. $|x|_1 = |b|_1^{\varrho}$

Pick $r/s \in \mathbb{Q}$, $r, s \in \mathbb{Z}$, $\varrho \leq \frac{r}{s}$

$\Rightarrow \quad |x|_1 \leq |b|_1^{\frac{r}{s}}$

$\Leftrightarrow |x^s \cdot b^{-r}|_1 \leq 1 \Leftrightarrow x^s \cdot b^{-r} \in \mathcal{O}_K$

$\Rightarrow |x^s \cdot b^{-r}|_2 \leq 1 \Leftrightarrow |x|_2 \leq |b|_2^{\frac{r}{s}}$

Similarly, $|x|_2 \geq |b|_2^{\frac{r}{s}} \quad \forall \frac{r}{s} \leq \varrho$

$\underset{\frac{r}{s} \to \varrho}{\Rightarrow} \quad |x|_1 = |x_2|$

2) $\Rightarrow$ 3) ✓ as subspace top. on

$\mathcal{O}_K$ is (x)-adic $\forall x \in m_K \setminus \{0\}$

3) $\Rightarrow$ 2) Note:

$m_K = \{x \in K \mid x^n \to 0,\ n \to \infty\}$

only depends on metric top. on K

$\Rightarrow \mathcal{O}_K = \{x \in K \mid x \cdot y \in m_K \ \forall y \in m_K\}$

depends only on the top. on K □

"$\subseteq$" ✓

"$\supseteq$" Assume $x \notin \mathcal{O}_K$, $x \in RHS$

$\Rightarrow |x| > 1$

$\Rightarrow y = x^{-1} \in m_K$ & $x \cdot y = 1 \notin m_K$

↯

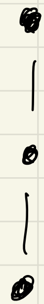
$\boxed{\mathbb{R}\times\mathbb{R}}$ lexicographic order

i.e. $(x,y)\geq(z,w)$

$\Leftrightarrow$ $x>z$

or $x=z$ & $y\geq w$

Spec $\mathcal{O}_K$



⚠ Ex: K any number field,

$\mathfrak{p}\subseteq\mathcal{O}_K$ max.

$\Rightarrow$ $\mathcal{O}_{K,\mathfrak{p}}$ completion of $\mathcal{O}_K$ w.r.t. $\mathfrak{p}$-adic top.

$K_{\mathfrak{p}}=\mathrm{Frac}(\mathcal{O}_{K,\mathfrak{p}})$ finite ext. of $\mathbb{Q}_p$, where $p\in\mathfrak{p}$

$K_{\mathfrak{p}}$ is a non-arch. valued field

& $\mathfrak{p} \mapsto K_{\mathfrak{p}}$

is a bijection between

$\operatorname{Spec} \mathcal{O}_K \setminus \{(0)\}$

& {completions of non-arch., non-trivial norms on K}

K any number field,

$K \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$

$\Rightarrow$ #{compl. of non-trivial, arch. norms on K, isom. to $\mathbb{R}$} $= r_1$

& #{ . . . — — — — to $\mathbb{C}$} $= r_2$

Let $(K, |\cdot|)$ be a complete, discretely valued non-arch. field

$U_K = \mathcal{O}_K^\times = \{ x \in K \mid |x| = 1 \}$

Let $\pi \in \mathcal{O}_K$ uniformizer, i.e. $(\pi) = m_K$

$$U_K^n := \{ x \in U_K \mid x \equiv 1 \mod \underbrace{(\pi)^n}_{m_K^n} \}$$

group of 1-units

$\mathcal{O}_K = \varprojlim_n \mathcal{O}_K/(\pi)^n$

Note: $U_K^0 = U_K$, $U_K^0/U_K^1 \simeq k^\times$,

$k = \mathcal{O}_K/m_K$ residue field

Let $n \geq 1$

Claim: $\varphi : U_K^n/U_K^{n+1} \xrightarrow{\sim} k$ depends on π

$$x \longmapsto \frac{x-1}{\pi^n} \mod m_K$$

Proof: φ well-dfd, if $x = \underbrace{1 + y \cdot \pi^n}$ for some $y \in \mathcal{O}_K$

$\Rightarrow \varphi(\bar{x}) \equiv y \mod m_K$

$\Rightarrow \varphi$ inj. + surj.

invertible in $\mathcal{O}_K$ for all $y \in \mathcal{O}_K$

by (π)-adic completeness of $\mathcal{O}_K$

We'll need slight strengthening of Hensel's lemma (actually, equivalent Tag 04GG, SP)

K complete, non-arch valued

$f \in K[x]$ is called primitive if $f \in \mathcal{O}_K[x]$ and $f \not\equiv 0 \mod m_K$

Prop (Hensel's lemma): $f \in K[x]$ primitive, $\bar{f} = g_0 \cdot h_0$ with $g_0, h_0 \in k[x]$

$(g_0, h_0) = 1$ in $k[x]$

Then $f = g \cdot h$ with $g, h \in \mathcal{O}_{K}[X]$,

$\deg g = \deg g_0$, $\deg h = \deg h_0$

& $\bar{g} \equiv g_0$, $\bar{h} \equiv h_0$

Moreover, g, h are uniquely det. by this up to mult. by an elt in $\mathcal{O}_{K}^{\times}$

Prf: Again approx., Details in Tian, Prop. 8.4.1. $\square$

If $g_0 = X - \alpha_0$, $\alpha_0 \in k$,

$(g_0, h_0) = 1$ is equiv. to

$h_0(\alpha_0) \neq 0$, i.e. $f'(\alpha_0) \neq 0$

$\Rightarrow$ $f = (X - \alpha) \cdot h$ with $\alpha \equiv \alpha_0 \bmod m_K$

in this case

$\Rightarrow$ This form of Hensel's lemma implies our previous

Corollary: $f(x) = \sum_{i=0}^{n} a_i x^i \in K[x]$

irred. of deg n

$\Rightarrow \quad \|f\| := \max\limits_{i=0,\dots,n} (|a_i|) = \max(|a_0|, |a_n|)$

Prf: wlog $\|f\| = 1$ (by multiplying with some $a \in K^\times$)

Pick j minimal, s.t. $|a_j| = \|f\| = 1$

$$\Rightarrow \quad \bar{f}(x) \equiv a_j x^j + \dots + a_n x^n$$

$$= x^j \cdot (a_j + \dots + a_n x^{n-j})$$

relatively prime if $0 < j < n$

as $a_j \not\equiv 0 \mod \mathfrak{m}_K$

$\Rightarrow$ $f = x^j \cdot h$, $\deg h = n-j$

Hensel's la

$\Rightarrow$ contradiction to irreducibility of f if $0 < j < n$

$\square$